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RF IMPULSE SPECTRUM ANALYZER

FIRST QUARTERLY PROGRESS REPORT

MAY 15, 1953 TO AUGUST 15, 1953

SIGNAL CORPS CONTRACT NO. DA-36-039 SC-52629

DEPT. OF THE ARMY PROJECT NO. 3-54-03-042

SIGNAL CORPS PROJECT NO. 2224B

PLACED BY

U.S. ARMY SIGNAL CORPS ENGINEERING LABORATORIES,

FORT MONMOUTH, N. J.

6. 3.5

Object: The design, development and construction of one (1) experimental RF impulse spectrum analyzer.

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ERRATA

Appendix A, Page 8-2, Equation (6)

Should be

$$e_{oi} = \frac{pAg_m}{C\left[p^2 + \frac{p}{CR} + \frac{1}{LC}\right]} = \frac{pAg_m}{C(p+\alpha)^2 + \beta^2}$$

$$= \frac{pAg_m}{C(p+\gamma) + (p+\Gamma)}$$

In all equations from equation (6) to the end of the chapter interpret G^n as being $g_m{}^n$, in other words, instead of being the n'th power of the gain, it should be the n'th power of the mutual conductance.

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PURPOSE

The purpose of this contract is the design, development and construction of an impulse spectrum analyzer continuous over the frequency range of 150 KC to 1 kilomegacycle, for use in the measurement of the absolute spectral intensity of very short, separated, repetitious pulses. Work by Lynmar to date would indicate that one possible solution of the problem is the use of a series of cascaded and synchronously tuned RLC shunt circuits with vacuum tube isolation between stages (hereinafter known as CASTAS). In order that details given in this report are clear to all, definitions of the various terms used will be given along with basic formulae and references.

References:

1. Signal Corps Specification SCL-1415 dated 12 December 1952 "Impulse Spectrum Measuring Equipment"
2. "Impulse Excitation of a Cascade of Series Tuned Circuits" by Samuel Sabaroff, Proceedings of I.R.E., Volume 32, No. 12, December 1944
3. "Frequency Analysis Modulation and Noise" by Goldman
4. "Transformation Calculus and Electrical Transients" by Goldman

A. Impulse

Mathematically an impulse is a function of infinite height and infinitesimal width such that the area under the impulse is some constant. Under these conditions the value for the area does not depend on the shape of the impulse. Practically a pulse may be considered to be an impulse if

its time duration is very much less than the time of the high frequency cut-off of the amplifier to which it is applied. (See reference 1, paragraph 3.2.2 and reference 4, page 100-103)

B. Spectral Intensity

Spectral intensity mathematically is simply twice the area under the volt time impulse curve and for a true impulse the spectral intensity is independent of frequency.

The spectral intensity of an unknown impulse may be measured by applying the impulse to a CASTA satisfying the approximation as made in reference 2 and noting the peak value of the transient, then applying a sine wave at the center frequency of the CASTA with such a magnitude as to make the output equal in amplitude to the peak value of the transient. If the impulse noise bandwidth for this CASTA is known, the spectral intensity may be calculated by dividing the impulse noise bandwidth (to be defined later) into the peak value of the input sine wave. (See reference 1, paragraph 3.2.2, reference 3, page 127 and reference 4, page 101)

C. Impulse Noise Bandwidth

The bandwidth as we normally think of it is taken quite arbitrarily at the 3 db points either side of the center frequency. For any given amplifier, if an impulse of known spectral intensity is applied to the input and the peak response determined, the impulse noise bandwidth may be calculated by dividing peak response by the spectral intensity. If the constants of the tuned circuit and amplifiers are known,

the impulse noise bandwidth, K, may be calculated from the formula:

$$K = \frac{(n-1)^{(n-1)} \;\varepsilon^{-(n-1)}}{2RC \;\underline{/n-1}}$$

(See page 4-1 for the definitions of the above parameters)

(See reference 1, paragraph 3.2.3)

D. <u>Response of "n-" Cascaded Series Tuned RLC Circuits from Impulse Excitation</u>

In reference 2, it is shown that there is a definite relationship between the amplifier constants, peak output and the value of the spectral intensity at the input. Although this analysis is for a series tuned circuit, by the principle of duality[1] it may then be extended without any approximations to a shunt tuned circuit. Therefore, all the information as given in reference 2 does hold for the case of a series of cascaded synchronously tuned shunt RLC circuits with vacuum tube isolation between stages.

[1] Transients in Linear Systems by Gardner and Barnes pgs 43-49.

ABSTRACT

The Proposed Lynmar Method for Determining Impulse Strength

It will be shown (Appendix A) that the impulse response of n- cascaded shunt tuned circuits is given approximately by the following equation:

$$v_n(t) = 2AG \frac{\alpha^n}{\lfloor n-1} t^{(n-1)} \varepsilon^{-\alpha t} \begin{matrix}\sin \\ \cos\end{matrix} (\omega t) \qquad (1)$$

The above parameters are defined on page 4-1.
If this response is rectified so that the carrier or high frequency component is effectively removed then the envelope function may be written as

$$v_n(t) = 2AG \frac{\alpha^n}{\lfloor n-1} t^{(n-1)} \varepsilon^{-\alpha t} \qquad (2)$$

In the above equations the value for RC has been replaced by its equivalent in alpha. In searching for some relationship between the above equation and some easily measurable characteristic it was found that when the envelope function itself was considered to be a driving pulse for a succeeding cascaded parallel tuned RLC circuit then a simple relationship existed between the initiating impulse and the response of this second cascaded set of tuned circuits.

Integrating the time function of the above equation with respect to "t" from zero to infinity, the following relationship is found to result.

$$\text{Area} = 2AG \qquad (3)$$

It is evident that the characteristics of the first set of tuned circuits have disappeared and that the only remaining

relationship is that of the originating impulse strength and the center frequency gain of the first amplifier. In effect an impulse in its entirety has been transferred from one time domain to another in which the time has been effectively stretched. This imposes certain requirements on the second amplifier. The most apparent requirement is that the effective duration of the output of the first set of tuned circuits shall be very small with respect to the period of the second set of tuned circuits. The output of the second set of tuned circuits will then have the normal form as shown in equation (2) above. Normal measuring techniques can then be applied to this output in order to determine the characteristics of the first initiating pulse.

The output of the second set of circuits can in turn be rectified and thence applied to a third set of circuits in which the output will also be of the form shown in equation (2) above. Obviously, this cannot be carried on indefinitely because the time domain will become too slow to contain very many isolated pulses per unit time. This means that the pulse repetition rate must be so adjusted that overlapping will not occur to any appreciable degree.

There are several specific problems that must be investigated in detail, and in fact is being done at the present, before serious construction on the final experimental model can be started.

1. <u>Rectifier Linearity</u>

Experimental work is being done in an effort

to determine the requirements for adequate linearity in detector response. It is evident that the accuracy of the actual measurement of impulse response will be greatly influenced by this rectifier characteristic.

2. Pulse Repetition Rate

Due to the greatly slowed down time domain that results from the scheme as described above, the pulse repetition rate and the effect of overlapping pulses must be investigated quite thoroughly. This is being done both analytically and in the laboratory so that limiting requirements might be determined.

3. Peak Value Measurement

A great deal of work is being done on methods for measuring the peak value of the responses of the various sets of circuits. It has not yet been determined what the best schemes for measuring peak response will be.

An Area Method for Measuring Impulse Strength

Upon close study of the proposed Lynmar method for determining impulse strength and examination of the derivation of equation (18), Appendix A, another method for measuring impulse strength presented itself. Equation (2), Appendix E, shows that the integration of the envelope of the response of the cascaded circuits is proportional directly to impulse strength. Instead therefore of measuring the peak value of the resulting envelope a determination of impulse strength can be effected by accurately performing this indicated integration. This idea has been discussed in some detail at a meeting with Signal Corps engineers. The merit of this scheme lies in the fact that the result is independent of the characteristics of the various tuned circuits provided however that they satisfy the requirements for the accuracy of equation (18), Appendix A.

The area of the envelope in volt seconds will ordinarily be quite small. Since the pulses are repeated it is thought that a scheme might be evolved whereby the repetitious nature of these pulses could be utilized in an additive fashion. For instance, an electronic integrator could be utilized in such a fashion that it integrated a predetermined number of pulses. This could be accomplished for example by means of a preset counter with a gate. The gate then could turn the integrator off and on for this predetermined number of pulses.

Another scheme for determining the area of the envelope of equation (18), Appendix A, would be simply to planimeter the trace as shown on a scope. This trace of course would be calibrated in terms of volts and seconds.

The Signal Corps engineers have been presented with a photograph of a CRT trace of the above mentioned output. The trace was planimetered and the correlation between spectral intensity by this method and the peak method was within 10%.

CONFERENCES

1. A formal conference between Lynmar Engineers and the engineers of the Signal Corps was held at the Signal Corps on June 18, 1953. In this conference the aims and objectives of this contract were examined and thoroughly discussed. Lynmar Engineers presented a proposed scheme for measuring impulse strength and stated that a period of time would be necessary for further study of this scheme so that its prospective use could be verified. Problems that had been encountered in the purchase of equipment were brought up at this time with the result that the Signal Corps engineers agreed to investigate the possibilities for the loan of some government equipment to Lynmar until such time as the procurement of equipment could be effected.

2. An informal conference between Lynmar Engineers and engineers of the Signal Corps was held on July 8, 1953. The Signal Corps were very helpful in their suggestions as to possible schemes and methods for measuring peak voltages. Several circuits were discussed and considered from the point of view of stability and reliability. The Signal Corps engineers demonstrated various pieces of equipment in their laboratories which proved of great interest and showed promise in eventual use for measurement purposes.

3. On 17 August 1953, a conference was held between Lynmar engineers and Signal Corps engineers in which Lynmar engineers

presented the results of a theoretical investigation of measurement techniques that might conceivably simplify the desired end equipment. This scheme involved the integration of the pulse envelope thereby, at least theoretically, eliminating the effects of any intervening tuned circuitry. It was deemed of such importance that it was decided to hold another conference to review the results of further study.

4. On 25 August 1953, another conference was held between Signal Corps engineers and Lynmar engineers for the purpose of reviewing and discussing the scheme for impulse measurement suggested by Lynmar engineers in the conference of 17 August 1953. Both Lynmar engineers and the Signal Corps engineers presented the results of their investigations and discussions brought forth the following items.

a. Detailed investigation of the effect of noise on the area method for measuring impulse strength.

b. Further study on the results of repeated pulses on the accuracy of measurement.

It was suggested by the Signal Corps engineers that the area method for measuring impulse strength might be of universal application and that some general formulation might indicate this to be true. It was decided therefore that further study would be given to this suggestion and the results presented at a later date.

FACTUAL DATA

List of Symbols

The following is a list of symbols with definitions, that will be used throughout this report.

CASTA	Cascaded and Synchronously Tuned Amplifier of n Stages
S_n	Spectral Intensity - output from n'th series of CASTA's e.g. input to first CASTA is S_o and its output S_1
f_o	Center frequency of a CASTA
α	$\frac{1}{2RC}$
αt	Damping time factor for a CASTA (use for voltage and/or area)
$(\alpha t)^1$	Incrementally increased damping time factor for a CASTA (in most cases 115% αt)
Ra_n	$\frac{\text{Envelope area integral-limits 0 to infinity}}{\text{Envelope area integral with limits t to infinity}}$ = area ratio for n stag
$R'a_n$	Value of area ratio for $(\alpha t)''$
Rv_n	$\frac{\text{Envelope maximum}}{\text{(Voltage at any time = t)}}$ = voltage ratio for n stages
$R'v_n$	Value of voltage ratio for $(\alpha t)'$ for n stages
Da_n	Derivative of area ratio for n stages
Dv_n	Derivative of voltage ratio for n stages
n	Number of tuned circuits
R	Total parallel resistance of tank circuit in ohms
L	Total parallel inductance of tank circuit in henries
C	Total parallel capacitance of tank circuit in farads
G	Gain at resonant frequency
g_m	Mutual conductance
K	Impulse noise bandwidth

M,C,B	Constants
B'	B + ΔB (in most cases 115% B)
A	Area of impulse (volt-seconds)
t	Time
Δf	$f_2 - f_1$ (difference between 3 db frequencies)
Q	Quality factor = $\frac{f_o}{\Delta f}$
a	Attenuation
Km_n	Maximum value of detected envelope of output of a CASTA when impulse excited
T	Period of repetition
ω	Resonant angular velocity, radians/sec.

The Investigations this Quarter have been directed Primarily to Determine the Properties of the Detected Output of a CASTA as a Result of Impulse Excitation.

The results of these investigations are given in some detail in Appendices A to G. A concise discussion of the material in the Appendices is given here.

Appendix A

"The response of "n" cascaded shunt synchronously tuned circuits (CASTA) to an impulse and to repeated impulses."

For a Single Impulse

Equation (18) of this appendix, namely

$$v_n = \frac{AG^n}{C^n 2^{(n-1)} \underline{/n-1}} t^{(n-1)} \varepsilon^{-\alpha t} \begin{matrix} \sin \\ \text{or} \\ \cos \end{matrix} (\omega t + K) \qquad (1)$$

is the approximate equation for this response

while $\frac{AG^n}{C^n 2^{(n-1)} \underline{/n-1}} \; t^{(n-1)} \varepsilon^{-\alpha t}$ is the envelope function (2)

Photographs of equation (1) are shown in Appendix F, fig. (1) and (2).

For Repeated Impulses

The output of a CASTA when excited by repeated impulses is

$$v_n(t) = \sum_{n=1}^{Q} C \left[t-(n-1)T\right]^{(n-1)} \varepsilon^{-\alpha\left[t-(n-1)T\right]}$$

where $C = \frac{2AG^n \alpha^n}{\underline{/n-1}}$

Appendix B

"Damping time factor (αt) for envelope decay"

In this appendix various formulae are derived from which the voltage damping time factor has been computed for different n. The meaning of the voltage damping time factor as used here is: that value of αt for which the envelope has decayed to some fraction of maximum taken as 0.01 (arbitrarily) for the examples shown.

This section represents a purely exploratory excursion into the question of overlap from the point of view of a usable engineering approximation.

The function, Rv_n, is shown to be a very rapidly varying one with respect to the voltage damping time factor. Comparison of Appendix B and Appendix C reveals that any

error based on the criterion of Appendix B will be of the approximate order of magnitude but that a slightly greater degree of latitude may be employed if the results of Appendix C are used.

Results of Appendix B are tabulated below.

The general expression for Rv_n, Dv_n are

$$Rv_n = \frac{\varepsilon^{(n-1)(B-1)}}{B^{(n-1)}}$$

$$Dv_n = Rv_n \left[\frac{(n-1)(B-1)}{B} \right]$$

"n"	αt	B	Rv_n	B'	$(\alpha t)'$	$R'v_n$	$\frac{R'v_n}{Rv\;\varepsilon\;n}$	Dv_n
3	9.78	4.89	100.044	5.6235	11.2470	329	329	159.17
4	11.7	3.9	101.1	4.485	13.455	386.14	381.94	225.747
5	13.48	3.37	101.529	3.876	15.504	439.125	432.512	285.6

Appendix C

The error in the maximum value of the envelope from repeated impulses is

$$\text{Error} = \sum_{m=2}^{Q} \left[1 + \frac{\alpha(m-1)T}{(n-1)} \right]^{(n-1)} \varepsilon^{-\alpha(m-1)T}$$

Appendix D

"Some information about an area method for measuring spectral intensity"

Equation (7), Appendix D, gives the value of the area of the detected output of a CASTA. Some tentative conclusions are given in Section 2.

Appendix E

"Damping time factor (αt) for area decay"

Various formulae are developed in this appendix from which the area damping time factor has been computed for different n. The meaning of the area damping time factor as used here is: that value of αt for which the area has decayed to some fraction of maximum, taken as 0.01 (arbitrarily) for the examples shown.

From the very definition of area it is obvious that the area damping time factor will be less than the voltage damping time factor. This implies that the repetition rate may be higher if an area method is used for measuring spectral intensity.

The general expressions for Ra_n, Da_n are

$$Ra_n = \frac{\varepsilon^{\alpha t} \lfloor\underline{n-1}}{(\alpha t)^{(n-1)} + (n-1)(\alpha t)^{(n-2)} + \ldots + \lfloor\underline{n-1}\,(\alpha t)^{n-(n-1)} + \lfloor\underline{n-1}}$$

$$Da_n = Ra_n \left\{ (\alpha t) - \left[\frac{(n-1)(\alpha t)^{(n-2)} + (n-1)(n-2)(\alpha t)^{(n-3)} + \ldots\ldots + \lfloor\underline{n-1}\,\alpha t^{\,n-(n-1)} + \lfloor\underline{n-1}}{(\alpha t)^{(n-1)} + (n-1)(\alpha t)^{(n-2)} + \ldots\ldots + \lfloor\underline{n-1}\,(\alpha t)^{n-(n-1)} + \lfloor\underline{n-1}} \right] \right\}.$$

Tabulated results of Appendix E

"n"	αt	Ra_n	$(\alpha t)'$	$R'a_n$	$R'a_n$ computed / Ra_n computed	$D'a_n$
3	8.006	99.531	9.66	274.097	275.39	818.97
4	10.05	100.36	11.558	310.584	309.47	981.94
5	11.6	99.682	13.34	341.807	342.899	1124.95

CONCLUSIONS

The work in this period has been of an exploratory nature in which detailed analysis has been initiated with respect to our problem as stated in Section 1. A definite scheme has not yet been crystallized, but as a result of the work done and work now in progress, it is expected that a program of construction will be initiated which will lead to a successful end item.

In Appendices A, B, and C, an analysis of the response of a CASTA is made for both single and repeated impulses. The error due to the repetitious nature of the impulses has been evaluated at the point of maximum voltage amplitude and tabulated in Section 10, Appendix C.

A preliminary study has been initiated for measuring the absolute spectral intensity by means of an area method as outlined in Section 11, Appendix D. This scheme shows great promise providing certain aspects of the relation between signal and noise is taken into consideration. Further study both theoretical and in the laboratory will be made in order to arrive at a proper evaluation of this method. Some of the preliminary analysis is described in Section 12, Appendix E.

For the purpose of experimental verification of theoretical derivations, a CASTA was built in the laboratory. This CASTA is described in detail in Section 13 Appendix F.

A number of photographs were made of various types of CRT traces due to repeated impulses with and without overlap. The results and discussion of these traces is described in Section 14, Appendix G.

PROGRAM FOR NEXT INTERVAL

The program for next interval will be to definitely crystallize a scheme for fulfilling the requirements of this contract. Full discussion will be held at Signal Corps Engineers as to the proper means for accomplishing this end.

IDENTIFICATION OF KEY PERSONNEL

Edward M. Beeler

Mr. Beeler, a consultant on this project, was one of the first to receive a first class radio operator's license (107) from the United States Department of Commerce. Mr. Beeler has been connected with some of the largest shipboard radio installations in the country numbering among them the ships Manhattan, Washington and America. He also outfitted the submarine Natilus for the Sir Hubert Wilkins' expedition.

During the war Mr. Beeler not only worked in research on airborne radar equipment at the Naval Research Laboratory at Anacostia, Washington, D. C., but also established radar repair shops and a school for radar training at Santa Barbara, California.

After the end of the war, Mr. Beeler went to work for station WPTZ in Philadelphia where he operated their video-audio and link transmitters. In 1947 Mr. Beeler began work on master antenna distribution systems. It was as an outgrowth of this work that Lynmar Engineers was conceived.

A real "old timer" in the field of electronics and electro-mechanical devices, Mr. Beeler's thirty-four years of technical experience provides immeasurable stature to the staff of this project.

Dr. Arsene N. Lucian, PhD Physics, Yale Graduate School, June 1914

Dr. Lucian, a consultant on this project, is well known in the field of physics, electronics and vacuum tube research. Dr. Lucian holds many patents and is the author of many outstanding technical papers. Dr. Lucian's contributions to the art of engineering have been invaluable.

A Professor of Physics at the University of Pennsylvania from 1924 to 1941, he also acted as consultant in research to many large corporations. Dr. Lucian in January 1943 was a regional coordinator for eastern Pennsylvania in charge of the Coordinator's Office of the Engineering, Science and Management War Training Program of the United States Office of Education.

A more complete biography was submitted with the letter proposal to the Signal Corps.

Samuel Sabaroff, B.S. in E.E., Drexel Institute of Technology, 1931
M.S. in E.E., Moore School, University of Pennsylvania, 1937

Mr. Sabaroff has been for more than twenty years a consultant in the field of electrical engineering where the problems have required a high order of theoretical, mathematical and experimental knowledge. Mr. Sabaroff has published very extensively in the better technical magazines and journals. A list of some of his publications follows:

"Radiation of Vertical Antennas", R/9, October, 1935

"A Voltage Stabilized High-Frequency Crystal Oscillator Circuit", I.R.E., May 1937

"Frequency Controlled Oscillators", Communications, February, 1939

"An Ultra-High Frequency Measuring Assembly", I.R.E., March, 1939

"Feedback Applied to Oscillator Control", Electronics, May, 1940

"System of Phase and Frequency Modulation", Communications, October, 1940

"Theory of Frequency Controlled Oscillators", Journal of Applied Physics, August, 1940

"Scanning Theory", J. M. P. E., May, 1941

"New System of Frequency Modulation", Communications, September, 1941

"Rounded Edge Capacitor Plates", Electronics, October, 1944

"Impulse Excitation of a Cascade of Series Tuned Circuits", I.R.E., December, 1944

"Link Coupled Coil Design", Communications, August, 1946

"Technique of Distortion Analysis", Electronics, June, 1948

Mr. Sabaroff is extremely familiar with impulse measurements having published on this in the IRE, December, 1944, "Impulse Excitation of a Cascade of Series Tuned Circuits" and having been previously connected with a contract concerning an impulse measurement R.F. amplifier for the United States Army Signal Corps Engineering Laboratories, Fort Monmouth, New Jersey.

The following is a brief list of some of the projects for which Mr. Sabaroff has been a consultant.

Date	Contract No.	Gov't Dept.	Brief Description
1948	AT-40-1GEN5[illegible]	U.S. Atomic Energy	Alpha Hand Monitor
1948	SC 2818-PH-49	Signal Corps	R. F. Coils
1950	N126S-4937(P)	U. S. Navy	Antenna Couplers
1950	1769	U. S. Dept.Agriculture	Antennas

Date	Contract No.	Gov't Dept.	Brief Description
1951	22096-PH-51-14	Signal Corps	Switch Assembly
1949	2262-PH-50-7	Signal Corps	R.F. Transformer
1951	18172-PH-51-13D	Signal Corps	Special Coils
1951	AF28(099)-365	3151st Electronic Group	Filters
1948	11085-PH-48-77	Signal Corps	BC 339 Radio Transmitter
1950	15939-PH-50-13	Signal Corps	Special Transmitters and Coils
1950	19624-PH-51	Signal Corps	AN/GRC-26 & AN/MRC-2 Amplifiers Dual Divensities Exciters
1948	1621-PH-49-2	Signal Corps	AN/TRC-3 & 4
1947	12702-PH-47-77	Signal Corps	CV-31/TRA-7 Dual Diversity Converter
1952	54-PH-52-91(1804)	Signal Corps	Radio Set/GRD-3
1952	3472-PH-52-06	Signal Corps	Radio Set AN/MRQ-2 ( )

Chico G. DeCoatsworth, B.S. in E.E., Pennsylvania Military College, June, 1950

Mr. DeCoatsworth is at present well on the way to obtaining a master's degree at the University of Pennsylvania where he is attending the Evening Graduate School. In addition, he is a state license instructor in radio and television theory, service and practice and is teaching at Radio Electronics Institute.

Mr. DeCoatsworth worked in a radar design and development laboratory at the Philco Corporation on one of the largest airborne radar sets for about a year. For approximately the next two years, he worked at the Frankford Arsenal as the electrical project engineer in charge of the bomb fuse research and development program.

APPENDIX A

Response of "n" Cascaded Shunt Synchronously Tuned Circuits (CASTA) to an Impulse and to Repeated Impulses

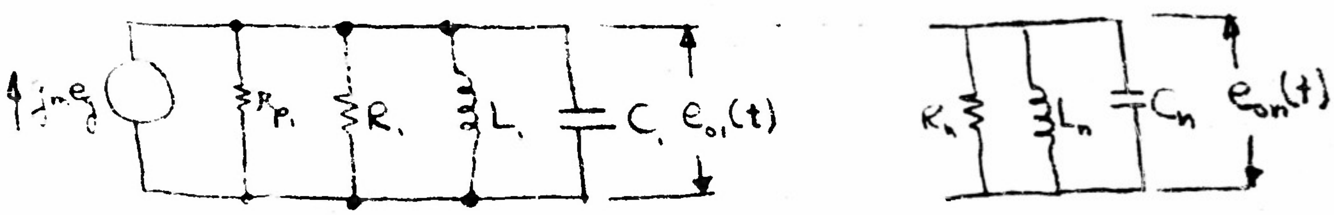


Figure 1

The Norton equivalent of one stage of a CASTA is shown in figure 1 for which the current equation is by Kirchoff's Law.

$$g_m e_g = i_{rp} + i_c + i_L + i_R \qquad (1)$$

We will neglect i_{rp} as being $\ll (i_R + i_c + i_L)$ at all frequencies.

$$\therefore\ g_m e_g = i_R + i_c + i_L \qquad (2)$$

Writing the differential equation for the above in terms of the output voltage gives

$$g_m e_g = \frac{e_{oi}}{R} + \frac{c\,de_{oi}}{dt} + \frac{1}{L}\int e_{oi}\,dt \qquad (3)$$

We shall assume constant circuit parameters and initial conditions at $t = o$ quiescent. Then

$$\frac{d}{dt} \equiv p \equiv j\omega \text{ and } \qquad dt \equiv \frac{1}{p} \equiv \frac{1}{j\omega} \qquad (4)$$

equation (3) becomes

$$e_{o1} = \frac{g_m e_g}{\frac{1}{R} + Cp + \frac{1}{pL}} = \frac{g_m e_g RpL}{p^2CLR + pL + R} \tag{5}$$

e_{o1} = output voltage of the first tuned circuit

where now e_g will be considered to be A, the Laplace transform of the impulse applied to the first tuned circuit and is a constant equal to the impulse strength. Then equation (5) becomes

$$e_{o1} = \frac{AGpL}{p^2CRL + pL + R} = \frac{ApG}{C(p + \gamma)(p + \Gamma)} \tag{6}$$

$$= \frac{ApG}{C\ (p + \alpha)^2 + \beta^2}$$

since $g_m R = G =$ Gain

where

$$\gamma = \alpha + j\beta$$

$$\Gamma = \alpha - j\beta$$

$$\alpha = \frac{1}{2RC}$$

$$\beta = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

For "n" such tuned circuits the output will be

$$e_{on}(p) = \frac{p^n G^n A}{C^n (p + \gamma)^n (p + \Gamma)^n} \tag{7}$$

$$= \frac{p^n G^n A}{C^n \left[(p + \alpha)^2 + \beta^2\right]^n}$$

which may be written

$$\frac{(C)^n e_{on}(p)}{A(p)^n G^n} = \left[(p + \alpha)^2 + \beta^2\right]^{-n} \tag{8}$$

The inverse Laplacian of the right hand side of equation (8) is*

$$L^{-1}\left[(p+\alpha)^2+\beta^2\right]^{-n} = \frac{t^{n-1}\,\epsilon^{-\alpha t}}{{}^{n}2^{n-1}\quad(n)}\sqrt{\frac{\pi\beta t}{2}}\;J_{n-\frac{1}{2}} \qquad (9)$$

where $n > 0$ and $J_{n-\frac{1}{2}}$ is a Bessel function of the first kind.

Using this and solving for $v_n(t)$ results in

$$v_n = \frac{AG^n}{C^n\beta^n 2^{n-1}\,\Gamma(n)}\;\frac{d^n}{dt^n}\left[t^{n-1}\epsilon^{-\alpha t}\sqrt{\frac{\pi\beta t}{2}}\;J_{n-\frac{1}{2}}(\beta t)\right] \qquad (10)$$

where p^n has been identified with the operation $\frac{d^n}{dt^n}$

and the validity of this may be justified by the initial value theorem. (See Theorem 15, Page 267, Transients in Linear Systems by Gardner & Barnes)

<u>Conversion of the Bessel Function to Circular Functions</u>

β in equation (10) may be real, zero, or imaginary, depending on whether the circuits are oscillatory, critically damped, or aperiodic. Only the oscillatory case for which β is real will be considered. The Bessel function in equation (10) may be rewritten in terms of circular functions.** Thus,

$$\sqrt{\frac{\pi\beta t}{2}}\;J_{n-\frac{1}{2}}(\beta t) = nPr(\beta t)\cos\left(\beta t-\frac{n\pi}{2}\right) - nQr(\beta t)\sin\left(\beta t-\frac{n\pi}{2}\right) \qquad (11)$$

where n is a positive integer and

*R.V. Churchill, "Modern Operational Mathematics in Engineering," transform No. 11, pg 294 and No. 57 pg. 298.

**S.A. Schelkunoff, "Electromagnetic Waves", pgs. 51-52.

$$nPr(\beta t) = \sum_{r=0}^{\infty} \frac{(-1)^r \, \underline{/n-1+2r}}{\underline{/2r} \;\; \underline{/n-2r-1} \;\; (2\;\;t)^{2r}} \tag{12}$$

$$nQr(\beta t) = \sum_{r=0}^{\infty} \frac{(-1)^r \, \underline{/n+2r}}{\underline{/2r+1} \;\; \underline{/n-2r-2} \;\; (2\;\;t)^{2r+1}} \tag{12a}$$

nPr and nQr are terminating series, since for sufficiently large values of r, the factorial of a negative integer appears in the denominator and such a factorial is equal to infinity.

nPr (βt) and nQr (βt) may be considered to be the amplitudes of "in-phase" and "quadrature" components.

In equations (12) and (12a), the highest power of "n" that occurs for a given "r" is the difference between the argument of the factorial involving "n" in the numerator and the argument of the factorial involving "n" in the denominator. Thus, in equation (12), for a given "r", the highest power of "n" is $(n-1+2r)-(n-2r-1) = 4r$ which is twice the power of (βt) in the denominator, and in equation (12a) the highest power of "n" is $(n+2r)-(n-2r-2) = (4r+2)$ which again is twice the power of (βt) in the denominator.

In equations (12) and (12a), the passage of time will make all terms negligible compared to the leading terms. Therefore as a first approximation, for

$$\beta t > n^2 \tag{13}$$

the leading terms in equations (12) and (12a) are

$$nPo\,(\beta t) = 1 \tag{14}$$

$$nQo\,(\beta t) = \frac{n(n-1)}{2\;\;t} \tag{14a}$$

Using equations (11), (14) and (14a) in equation (10) results in

$$v_n = \frac{AG^n}{C^n \beta^n 2^{n-1} \lfloor n-1} \frac{d^n}{dt^n} \left\{ t^{n-1} \varepsilon^{-\alpha t} \cos\left(\beta t - \frac{n\pi}{2}\right) \right. \quad (15)$$

$$\left. -\frac{n(n-1)}{2 \quad t} \sin\left(\beta t - \frac{n\pi}{2}\right) \right\}$$

The sine and cosine terms in equation (15) can be combined into a single cosine function with a modulus and a phase angle.

The modulus, $M = \sqrt{1 + \frac{n^2 (n-1)^2}{4(\quad t)^2}}$

The phase angle, $\gamma = \tan^{-1} \frac{n(n-1)}{2\beta t}$

For γ small, (the smallness of γ will be justified later), and using the binomial expansion as a first approximation, M and γ can be written

$$M \approx 1 + \frac{n^2 (n-1)^2}{8(\beta t)^2}$$

$$\gamma = \tan^{-1} \frac{n(n-1)}{2\beta t} \approx \sin^{-1} \frac{n(n-1)}{2\beta t} \approx \cos^{-1} 1 - \frac{n^2(n-1)^2}{8(\beta t)^2}$$

If it is assumed that the circuits are sufficiently oscillatory so as to make $\alpha \ll \beta$, the value of β approaches the fundamental angular resonant frequency of the circuits, which will be called ω_0. Thus,

$$\beta = \sqrt{\frac{1}{LC} - \frac{1}{\alpha^2}} \approx \frac{1}{LC} \approx \omega_0$$

Equation (15) may now be written, approximately,

$$v_n = \frac{AG^n}{C^n \; n_2^{\,n-1} \lfloor n-1} \frac{d^n}{dt^n} \quad (16)$$

$$\left\{ t^{n-1} \varepsilon^{-\alpha t} \left[1 + \frac{n^2(n-1)^2}{8(\omega t)^2} \right. \cos\left(\omega t + \gamma - \frac{n\pi}{2}\right) \right\}$$

Assumption of Negligibility of the Quadrature Component of the Modulus

The term $\frac{n^2(n-1)^2}{8(\omega t)^2}$ in equation (16) will be called the quadrature component of the modulus. Of interest is the differentiation in equation (16). Assume arbitrarily for a moment that time t_o has elapsed from $t = 0$ such that

$$\frac{n^2(n-1)^2}{8(\omega t_o)^2} < \frac{1}{100} \qquad (17)$$

Then, this term can be neglected since it is small compared to one. This results in the simplification of the derivative to

$$\frac{d^n}{dt^n}\left[t^{n-1} \varepsilon^{-\alpha t} \cos\left(\omega t + \gamma - \frac{n\pi}{2}\right)\right]$$

Relation (17) establishes a certain design criterion for the Q of each stage of the amplifier with "n" single tuned circuits.

When "n" is fixed, relation (17) establishes the number of cycles of carrier necessary for (17) to be valid. Thus, solving (17) for ωt_o, and say $n = 5$,

$$(\omega t_o)^2 > \frac{100n^2(n-1)^2}{8}$$

$$\omega t_o > 70.7$$

Let N = number of cycles of carrier, or the number of 2π radians required for (17) to be valid, i.e.,

$$t_o = N(2\pi)$$

Then,

$$N(2\pi) > 70.7$$

$$N > 11.3$$

say $\quad N > 12$ cycles

This means that when $n = 5$, at least 12 cycles of carrier must elapse before relation (17) is satisfied.

Constancy of the Phase Angle

A further simplification results if γ can be considered constant when taking successive derivatives of $t^{n-1}\varepsilon^{-\alpha t}$ $\cos\left(\omega t + \gamma - \frac{n\pi}{2}\right)$.

It may be seen that as ωt increases, γ approaches zero at a very slow rate compared to the rate of rise of ωt, so that γ may be considered constant with respect to ωt as far as the differentiation is concerned.

The approximate derivative is now

$$\frac{d^n}{dt^n}\left[t^{n-1}\varepsilon^{-\alpha t}\cos(\omega t + K)\right]$$

where $\quad K = \gamma + \frac{n\pi}{2} = \text{a constant}$

Constancy of the Modulus $t^{n-1}\varepsilon^{-\alpha t}$

The resulting approximate derivative in paragraph 4 above involves combination of products of successive derivatives of $(t^{n-1}\varepsilon^{-\alpha t})$ and $\cos(\omega t + K)$. It can be shown that the amplitude of the first term in the expansion of

$$\frac{d^n}{dt^n}\left[t^{n-1}\varepsilon^{-\alpha t}(\cos(\omega t + K)\right], \text{ i.e. } (t^{n-1}\varepsilon^{-\alpha t}\frac{d^n}{dt^n}\cos(\omega t + K),$$

is large compared to the remainder of the terms and therefore, a still further simplification results. This amounts to considering $(t^{n-1}\varepsilon^{-\alpha t})$ constant with respect to $\cos(\omega t + K)$ in the derivative.

This, of course, is the same procedure as is followed in the case of a modulated carrier.

The approximate derivative now becomes

$$t^{n-1} \mathcal{E}^{-\alpha t} \frac{d^n}{dt^n} \cos(\omega t + K)$$

The amplitude of the n'th derivative of cos $(\omega t + K)$ is ω^n, which will cancel the ω^n in denominator of equation (16). Hence equation (16) becomes

$$v_n = \frac{AG^n}{C^n 2^{n-1} \underline{/n-1}} t^{n-1} \mathcal{E}^{-\alpha t} \begin{matrix}\sin \\ \text{or} \\ \cos\end{matrix} (\omega t + K) \qquad (18)$$

Equation (18) is a sinusoid of varying amplitude.

The envelope function is

$$v_n(t) = \frac{AG^n}{C^n 2^{n-1} \underline{/n-1}} t^{n-1} \mathcal{E}^{-\alpha t} \qquad (19)$$

The result as obtained in equation (19) for equation (7) may have been obtained by applying the calculus of Residues.[1]

Consider

$$\frac{Ap^n G^n}{(p + \gamma)^n (p + \Gamma)^n C^n} = e_{on}(p) \qquad (20)$$

In this case $e_{on(t)}$ is the solution in the time domain in terms of the real variable t from the initial integro-differential equation in the frequency domain in terms of the complex variable p.

Equation (20) has n'th order poles at $p = -\gamma$ and $p = -\Gamma$. Following the notation as given in reference 1(a)

[1](a) Complex Variable and Operational Calculus by McLachlan

(b) Transients in Linear Systems by Gardner and Barnes

p. 49 paragraph 3.241.

Let the residue at $p = -\gamma$ be b_1 and $p = -\Gamma$ be b_2. Then the time solution to equation (20) may be obtained by completing the indicated operations

$$b_1 + b_2 = \frac{AG^n}{\underline{/n-1}} \left\{ \frac{d^{(n-1)}}{dt^{(n-1)}} \left[\frac{\varepsilon^{pt}}{(p+\Gamma)^n} \right]_{p=-\gamma} + \frac{d^{(n-1)}}{dt^{(n-1)}} \left[\frac{\varepsilon^{pt}}{(p+\gamma)^n} \right]_{p=-\Gamma} \right.$$

and by consistent• approximations.

Observe that in the differentiation t is a constant.

For a finite series of repeated impulses the Laplace transform takes the form of a finite series

$$e_g = A \sum_{m=1}^{Q} \varepsilon^{-p(m-1)T} = \frac{1}{\left[1-\varepsilon^{-pT}\right]}$$

If this is used to operate on the transfer function of a CASTA the following results by the use of the linearity theorem[1]

$$e_n(t) = \sum_{m=1}^{Q} L^{-1} \frac{p^n G^n \varepsilon^{-p(m-1)T}}{(p+\gamma)^n (p+\Gamma)^n}$$

from which as before by the Calculus of Residues

$$b_1 + b_2 = \sum_{m=1}^{Q} \frac{AG^n}{\underline{/n-1}} \left\{ \frac{d^{(n-1)}}{dt^{(n-1)}} \left[\frac{\varepsilon^{p\left[t-(m-1)T\right]}}{(p+\Gamma)^n} \right]_{p=-\gamma} + \right.$$

$$\frac{d^{(n-1)}}{dt^{(n-1)}} \left[\frac{\varepsilon^{p\left[t-(m-1)T\right]}}{(p+\gamma)^n} \right]_{p=-\Gamma}$$

Here as before the quantity $\left[t-(m-1)T\right]$ is constant in the differentiation. Therefore if the same approximations are made the solution for repeated pulses will be

[1] Transients in Linear Systems by Gardner and Barnes

$$v_n(t) = \sum_{m=1}^{Q} C \left[t-(m-1)T \right]^{(n-1)} \varepsilon^{-\alpha\left[t-(m-1)T\right]}$$

where $\quad C$ is $\dfrac{2AG^n \alpha^n}{\underline{|n-1}}$

APPENDIX B

The following represents a study of the detected envelope function with attention directed to the maximum voltage and to the time for the voltage to decrease to some fraction of the maximum for various values of n.

The approximate expression for the detected output voltage of a CASTA of n stages as a function of time, $v_n(t)$, is

$$v_n(t) = C\, t^{(n-1)}\, \varepsilon^{-\alpha t} \qquad (1)$$

where $C = \dfrac{2AG\,\alpha^n}{\lfloor n-1}$

The maximum value of $v_n(t)$ occurs at $t = \dfrac{(n-1)}{\alpha}$ and is equal to

$$E_{m_n} = C\left[\frac{n-1}{\alpha}\right]^{(n-1)} \varepsilon^{-(n-1)} \qquad (2)$$

Now let

$$Rv_n = \frac{E_{m_n}}{v_n} = \frac{C\left[\frac{n-1}{\alpha}\right]^{-(n-1)}}{C\, t^{(n-1)}\, \varepsilon^{-\alpha t}} \qquad (3)$$

and let $t = B\,\dfrac{(n-1)}{\alpha}$

$$\therefore\ Rv_n = \frac{\left[\frac{n-1}{\alpha}\right]^{(n-1)} \quad -(n-1)}{\left[\frac{B(n-1)}{\alpha}\right]^{(n-1)} \varepsilon^{-B(n-1)}} \qquad (4)$$

$$= B^{(1-n)}\, \varepsilon^{(n-1)\,[B-1]}$$

Evaluation of B for $n = 3$ at an arbitrary value of Rv_3

Arbitrarily let $Rv_n = 100$ for $n = 3$

$$\text{Then } Rv_3 = 100 = B^{(1-n)}\, \varepsilon^{(n-1)\,(B-1)} \qquad (5)$$

Let $100 = \varepsilon^{(1-n)M}$

$$\therefore\ Rv_3 = \varepsilon^{(1-n)M} = B^{(1-n)}\, \varepsilon^{(n-1)\,(B-1)} \qquad (6)$$

Taking the natural logarithm of both sides of this transcendental equation we get

$$(1-n)M = (1-n)\ln B-(1-n)(B-1)$$

$$M - 1 + B - \ln B = 0 \tag{7}$$

Now for $Rv_3 = 100$

$$\ln 100 = (1-n)M = 4.60517 \text{ and so } M = -2.30258$$

$$\therefore \quad -3.30258 + B - \ln B = 0 \tag{8}$$

Using Newton's Method[1] to approximate the root of this equation

$$\text{let } y = B - \ln B - 3.30258 = f(B) \tag{9}$$

Assume $f(B) = f(3.8) = 0$ is a solution (as a first approximation)

By Newton's Method a second approximation will be:

$$B' = B - \frac{f(B)}{f'(B)} = 3.8 - \frac{0.83758}{-0.9737} = 4.661 \tag{10}$$

where $f'(B)$ is the first derivative of equation (9)

Now by successive applications of the above method we arrive at the value $B = 4.89$

As a check

$$Rv_3 = \frac{\varepsilon^{(n-1)(B-1)}}{B^{(n-1)}} = \frac{2392.27482}{23.9121} = \underline{\underline{100.04}} \tag{11}$$

$$t = B\left[\frac{n-1}{\alpha}\right] = 4.89\,\frac{2}{\alpha}$$

$$\text{or } \alpha t = 9.78$$

[1]See p. 215 article 130 Elements of Calculus by Granville, Smith and Longley or any text on Calculus.

Similarly for $n = 4$ and $n = 5$ we get

$$Rv_4 = 101.1 \text{ for } B = 3.9, \quad \alpha t = 11.7 \tag{12}$$

$$Rv_5 = 101.529 \text{ for } B = 3.37, \quad \alpha t = 13.48 \tag{13}$$

It is of interest to see what the increase in Rv_n will be if αt is increased by 15%.

Using the general form as given in equation (4) let B be increased by 15% which is the same as increasing αt by 15% since $\alpha t = B(n-1)$

$$R'v_n = \{(1.15)(B)\}^{1-n} \; \varepsilon^{(n-1)(B-1)} \tag{14}$$

$$R'v_3 = 329 \tag{15}$$

$$R'v_4 = 386.14 \tag{16}$$

$$R'v_5 = 439.125 \tag{17}$$

Therefore increasing αt by 15% has produced:

for $n = 3$, $\dfrac{R'v_3}{Rv_3} = \dfrac{329}{100.04} = 329\%$ increase in Rv_3 (18)

for $n = 4$, $\dfrac{R'v_4}{Rv_4} = 381.94\%$ increase in Rv_4 (19)

for $n = 5$, $\dfrac{R'v_5}{Rv_5} = 432.5\%$ increase in Rv_5 (20)

The derivative, Dv_n, of equation (4) with respect to B is

$$Dv_n = \left.\frac{\partial Rv_n}{\partial B}\right|_{n = K} = \frac{\partial}{\partial B}\left(\varepsilon^{(n-1)(B-1)} \; B^{(1-n)}\right) \tag{21}$$

$$= \frac{\varepsilon^{(n-1)(B-1)}}{B^n} (n-1)(B-1) \tag{22}$$

$$= \varepsilon^{(n-1)(B-1)} B^{(1-n)} \left|\frac{(n-1)(B-1)}{B}\right|$$

but $Rv_n = \varepsilon^{(n-1)(B-1)} \; B^{(1-n)}$

equation (22) becomes

$$Dv_n = Rv_n \left[\frac{(n-1)(B-1)}{B} \right] \tag{23}$$

In general $n > B > 1$ for $n > 3$

$$\therefore \; Dv_n > Rv_n \quad \text{for } n > 3$$

Evaluating (23) for $n = 3, 4, 5$ with their respective B's

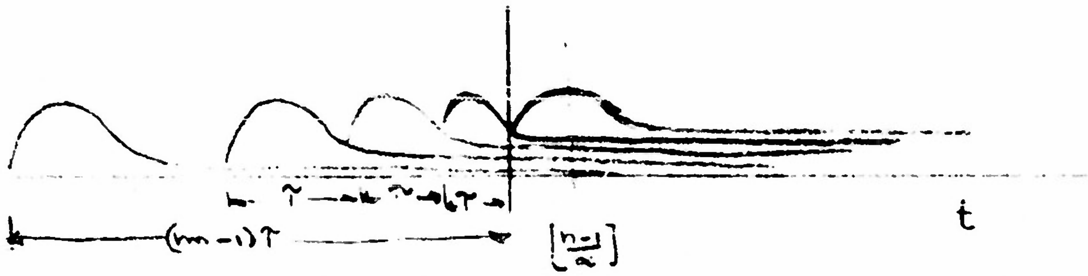
$$Dv_3 = 159.17 \tag{24}$$

$$Dv_4 = 225.7474 \tag{25}$$

$$Dv_5 = 285.608 \tag{26}$$

APPENDIX C

Repeated Pulses



Consider a finite series of detected pulses from a CASTA. Since we are interested in the voltage at some time, t, we may write the following summation from the above or from Appendix A.

$$e = \sum_{m=1}^{Q} \left[t + (m-1)T\right]^{(n-1)} \varepsilon^{-\alpha\left[t + (m-1)T\right]} \qquad (1)$$

$$= t^{(n-1)} \varepsilon^{-\alpha t} + \sum_{m=2}^{Q} \left[t + (m-1)T\right]^{(n-1)} \varepsilon^{-\alpha\left[t + (m-1)T\right]}$$

We are interested in equation (1) at t in the neighborhood of $(\frac{n-1}{\alpha})$

Equation (1) becomes

$$e = \left(\frac{n-1}{\alpha}\right)^{(n-1)} \varepsilon^{-(n-1)} + \sum_{m=2}^{Q} \frac{\left[\left(\frac{n-1}{\alpha}\right) + (m-1)T\right]^{(n-1)}}{\varepsilon^{+\left[(n-1) + (m-1)T\right]}} \qquad (2)$$

$$= \left(\frac{n-1}{\alpha}\right)^{(n-1)} \varepsilon^{-(n-1)} \left[1 + \sum_{m=2}^{Q} \left[1 + \frac{\alpha(m-1)T}{(n-1)}\right]^{(n-1)} \varepsilon^{-\alpha(m-1)T}\right]$$

but $Em_n = \left(\frac{n-1}{\alpha}\right)^{(n-1)} \varepsilon^{(n-1)}$ then (2) becomes

$$C = Em_n \left[1 + \sum_{m=2}^{Q} \left[1 + \frac{\alpha(m-1)T}{(n-1)}\right]^{(n-1)} \varepsilon^{-\alpha(m-1)T}\right]$$

where $\sum_{m=2}^{Q} \left[1 + \frac{\alpha(m-1)T}{(n-1)}\right]^{(n-1)} \varepsilon^{-\alpha(m-1)T}$

is the error for repeated pulses.

The above derivation does not take phase angles into consideration however, the maximum error for a given value of T will occur when the individual voltages are exactly in phase, therefore this error expression gives an approximation under the worst conditions.

Now we may specify that the error $\leq 1\%$ or we may examine the error for values of αt. It will be more enlightening here to compare the error due to (αT)'s somewhat less than the (αt)'s given in Appendix B.

$$\text{Error} = \sum_{m=2}^{Q} \left[1 + \frac{(m-1)T}{(n-1)}\right]^{(n-1)} \varepsilon^{-\alpha(m-1)T}$$

n	T	m	Error
3	8	2	0.0083866
3	8	3	0.00000911535
3	8	4	60.4×10^{-10}
4	10	2	0.0036857
4	10	3	0.00000092882
5	10	2	0.0068128
5	10	3	0.00000267

APPENDIX D

Some Information about an Area Method for Measuring Spectral Intensity

Consider equation (19), Appendix A.

$$v_n(t) = \frac{2AG\,\alpha^n}{\underline{/n-1}}\, t^{(n-1)} \varepsilon^{-\alpha t} = Ct^{(n-1)}\, \varepsilon^{-\alpha t} \qquad (1)$$

where $\frac{1}{2RC}$ has been replaced by α.

The general area integral is

$$C\int_{t_1}^{t_2} t^{(n-1)} \varepsilon^{-\alpha t}\, dt \qquad \text{where } t_2 > t_1 \qquad (2)$$

the time function may be integrated by parts viz.

$$\int u\,dv = uv - \int v\,du \qquad (3)$$

let $u = t^{(n-1)}$

$du = (n-1)t^{(n-2)}$

$dv = \varepsilon^{-\alpha t} dt$

$v = -\dfrac{\varepsilon^{-\alpha t}}{\alpha}$

Equation (2) becomes by equation (3)

$$= -\left.\frac{t^{(n-1)}\varepsilon^{-\alpha t}}{\alpha}\right|_{t_1}^{t_2} + \frac{(n-1)}{\alpha}\int_{t_1}^{t_2} t^{(n-2)}\, \varepsilon^{-\alpha t} dt \qquad (4)$$

Integrating by parts again equation (4) becomes

$$= -\left.\frac{t^{(n-1)}\varepsilon^{-\alpha t}}{\alpha}\right|_{t_1}^{t_2} - \left.\frac{(n-1)t^{(n-2)}\varepsilon^{-\alpha t}}{\alpha^2}\right|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{(n-1)(n-2)t^{(n-3)}\varepsilon^{-\alpha t}}{\alpha^2} \qquad (5)$$

Now we have established the pattern as follows:

$$\int_{t_1}^{t_2} t^{(n-1)} \varepsilon^{-\alpha t} dt = -\varepsilon^{-\alpha t} \left[\frac{t^{(n-1)}}{\alpha} + \frac{(n-1)t^{(n-2)}}{\alpha^2} + \frac{(n-1)(n-2)t^{(n-3)}}{\alpha^3} + \ldots\ldots + \frac{\lfloor n-1}{\alpha^n} \right] \Bigg|_{t_1}^{t_2} \tag{6}$$

$$= -t^n \varepsilon^{-\alpha t} \left[\frac{1}{\alpha t} + \frac{(n-1)}{(\alpha t)^2} + \frac{(n-1)(n-2)}{(\alpha t)^3} \right] \Bigg|_{t_1}^{t_2} - \left[\frac{\lfloor n-1 \, \varepsilon^{-\alpha t}}{\alpha^n} \right] \Bigg|_{t_1}^{t_2} \tag{7}$$

we observe that if the limits are $t_1 = 0$

$t_2 = \infty$

$$\frac{2AG\alpha^n}{\lfloor n-1} \int_0^{\infty} t^{(n-1)} \varepsilon^{-\alpha t} = \frac{2AG\alpha^n}{\lfloor n-1} \left[\frac{\lfloor n-1}{\alpha^n} \right] = 2AG \tag{8}$$

At no time in this discussion have we considered the validity of equation (1) at $\frac{(n-1)}{\alpha} > t > 0$.

A discussion of the implications of equation (8) will be found in Section 2.

APPENDIX E

The detected envelope of the output of a CASTA as a result of impulse excitation is given approximately by:

$$v_n(t) = \frac{2AG\alpha^n}{\underline{\lfloor n-1}}\, t^{(n-1)}\, \varepsilon^{-\alpha t} = Ct^{(n-1)}\, \varepsilon^{-\alpha t} \quad (1)$$

The integral of the area of this curve from $t = 0$ to $t = \infty$ is

$$\int_0^\infty v_n(t)dt = \int_0^\infty Ct^{(n-1)}\varepsilon^{-\alpha t}dt = \frac{C\underline{\lfloor n-1}}{\alpha^n} = 2AG = SG \quad (2)$$

Here it is tacitly assumed that equation (1) is valid in the region $\left(\frac{n-1}{\alpha}\right) \geq t > 0$

The integral of equation (1) from t to ∞ is

$$\int_t^\infty v_n(t)dt = C\left[\frac{1}{\alpha t} + \frac{(n-1)}{(\alpha t)^2} + \frac{(n-1)(n-2)}{(\alpha t)^3}\ \frac{+\ldots}{+\ldots} \quad (3)\right.$$

$$\left.\frac{\ldots+}{\ldots+}\ \frac{(n-1)(n-2)(n-3)\ldots\ldots[n-(n-2)]}{(\alpha t)^{(n-1)}} + \frac{\underline{\lfloor n-1}}{\alpha^n}\right]$$

The interest here is in the value of

$$Ra_n = \frac{\int_0^\infty v_n(t)dt}{\int_t^\infty v_n(t)dt} \quad (4)$$

from which a value of (αt) may be derived for any value of Ra_n.

The general form of Ra_n is

$$Ra_n = \frac{\varepsilon^{\alpha t}\underline{\lfloor n-1}}{(\alpha t)^{(n-1)} + (n-1)(\alpha t)^{(n-2)} + (n-1)(n-2)(\alpha t)^{(n-3)} + \ldots} \quad (5)$$

$$\frac{}{\ldots + \underline{\lfloor n-1}\,(\alpha t)^{n-(n-1)} + \underline{\lfloor n-1}}$$

For $n = 3$ $Ra_3 = 100$ (arbitrarily) equation (5) becomes

$$Ra_n = Ra_3 = \frac{2\,\epsilon^{+\alpha t}}{(\alpha t)^2 + 2\alpha t + 2} = 100 \qquad (6)$$

or $$(\alpha t)^2 + 2\alpha t + 2 - \frac{\epsilon^{\alpha t}}{50} = 0$$

We may use Newton's Method[1] to evaluate this transcendental equation, viz. assume $\alpha t = 8.4$

$$(\alpha t)' = \alpha t - \frac{f(\alpha t)}{f'(\alpha t)} = 8.4 - \frac{0.419}{-728.296} = \underline{\underline{8.400575}} \qquad (7)$$

For practical purposes let $\alpha t = 8.4$

As a check

$$Ra_3 = \frac{2\,\epsilon^{\alpha t}}{(\alpha t)^2 + 2\alpha t + 2} = \frac{8894.1336}{89.36} = \underline{\underline{99.531}} \qquad (8)$$

$\therefore$ $\alpha t = 8.4$ for $Ra_3 = 100$

<u>Similarly</u>

$$Ra_4 = 100.36, \quad t = 10.05 \qquad (9)$$

$$Ra_5 = 99.682, \quad t = 11.6 \qquad (10)$$

It is of interest to observe what the change in Ra_n, $R'a_n$, will be when (αt) is increased by 15%, $(\alpha t)'$.

Using equation (5)

$$R'a_3 = 274.097, \quad (\alpha t)' = 9.66 \qquad (11)$$

$$R'a_4 = 310.564, \quad (\alpha t)' = 11.588 \qquad (12)$$

$$R'a_5 = 341.807, \quad (\alpha t)' = 13.34 \qquad (13)$$

[1] See p. 215 article 130 Elements of Calculus by Granville, Smith and Longley or any text on Calculus.

Thus an increase in αt of 15% has produced

$$\frac{R'a_3}{Ra_3} = 275.39\% \text{ increase in } Ra_3 \qquad (14)$$

$$\frac{R'a_4}{Ra_4} = 309.47\% \text{ increase in } Ra_4 \qquad (15)$$

$$\frac{R'a_5}{Ra_5} = 342.899\% \text{ increase in } Ra_5 \qquad (16)$$

The derivative of Ra_n with respect to (αt)

$$Da_n = \frac{\partial}{\partial(\alpha t)}\Big|_{n=K} = \frac{\partial}{\partial(\alpha t)} \text{ (equation 5)} \qquad (17)$$

leads directly to the general form for Da_n

$$Da_n = Ra_n \left\{ (\alpha t) - \left[\frac{(n-1)(\alpha t)^{(n-2)} + (n-1)(n-2)(\alpha t)^{(n-3)} + \ldots}{(\alpha t)^{(n-1)} + (n-1)(\alpha t)^{(n-2)} + (n-1)(n-2)(\alpha t)^{(n-3)} + \ldots} \right.\right.$$

$$\frac{\ldots\ldots + (n-1)(n-2)(n-3)(\alpha t)^{(n-4)} + \ldots\ldots}{\ldots\ldots + \underline{/n-1}\,(\alpha t)^{\,n-(n-1)} + \underline{/n-1}}$$

$$\left.\left. \frac{\ldots\ldots + \underline{/n-1}\,(\alpha t)^{\,n-(n-1)} + \underline{/n-1}}{} \right] \right\} \qquad (18)$$

where Ra_n is as given in equation (5).

By straightforward computation we arrive at the following Da_n's evaluated at their respective (αt)'s.

$$Da_3 = 818.97 \qquad (19)$$

$$Da_4 = 981.94 \qquad (20)$$

$$Da_5 = 1124.95 \qquad (21)$$

APPENDIX F

13.7 KC CASTA (DATA)

For the purposes of experimental verification of theoretical derivations, a CASTA was built in the laboratory. The response curve is given on page 13-2. Data on this amplifier is as follows:

Tubes - (4) 6AC7 amplifiers

(1) 6J6 cathode follower (50 ohm)

$$\alpha = \frac{1}{2RC} = 1.098 \times 10^4$$

n = 5 (number of tuned circuits)

Gain = 3.16×10^4 (input grid to plate of last amplifier)

K = (impulse noise bandwidth) = 2.01 KC (Calculated)

$$Q = \frac{13.8}{2.01} = 6.86 = \frac{f_o}{K}$$

C = 0.0065 ufd.

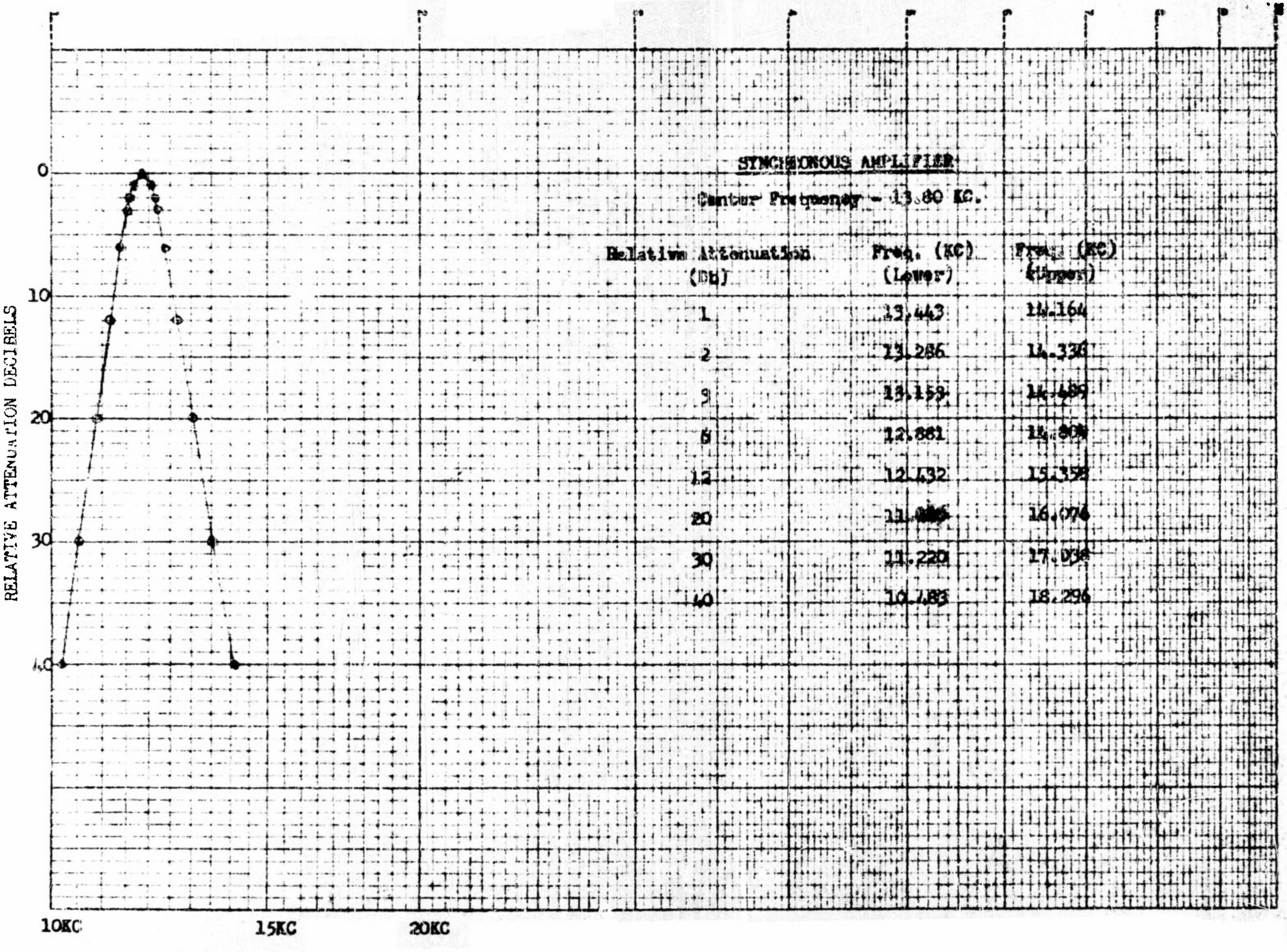
R = 7K

L = 19.4 mh.

$$\beta = 8.9 \times 10^4 = \frac{1}{\sqrt{LC}}$$

$$f_o = \frac{1}{2\pi\sqrt{LC}} = 14.16 \text{ KC (Calculated)}$$

f_o = 13.8 KC (Measured)



SYNCHRONOUS AMPLIFIER

Center Frequency - 13.80 KC.

Relative Attenuation (Db)	Freq. (KC) (Lower)	Freq. (KC) (Upper)
1	13.443	14.164
2	13.286	14.338
3	13.159	14.489
6	12.881	14.[illegible]
12	12.432	15.358
20	11.[illegible]	16.074
30	11.220	17.034
40	10.783	18.296

APPENDIX C

Reproduction of CRT Traces on the Response of a CASTA to Repeated Impulses with and without Overlap

The form of equation (18) Appendix A is shown quite clearly in figure 1. This figure is a trace of an impulse excited cascade of tuned circuits. The repetition rate has been so chosen that the pulse response has died down to a negligible value before the onset of the next exciting impulse. The exciting pulse occurs at the extreme left hand portion of the trace.

In figure 2, the repetition rate has been increased so that there is a small amount of overlap before the occurence of the next impulse.

Figure 3 clearly shows the effect of overlap. The impulses occur at such a rate that phase opposition takes place at the overlap points, thus creating a minimum as indicated.

In figure 4 however, the pulse repetition rate has been adjusted slightly so that phase addition takes place.

The trace shown in figure 5 shows the effect of a still further increase in repetition rate. It must be remembered that the exciting impulse takes place at the extreme left hand side of the trace and as is shown phase opposition occurs in such a manner as to create the minimum indicated.

In figure 6 however, the repetition rate has again been adjusted slightly so that the pulses overlap in an additive phase.

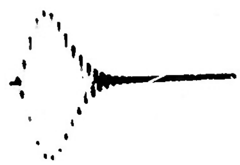


Fig. 1

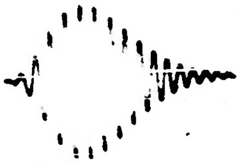


Fig. 2


Fig. 3


Fig. 4


Fig. 5


Fig. 6